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Detecting the Occurrence of an Event by FM Through Noise

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The occurrence of an event at a random time τ is signaled through white noise by an FM *signal whose modulation $h(t-\tau)$ is a causal pulse triggered at τ. Nonlinear filtering is used to find exact expressions for the chance that $\tau > t$, and the expectation of τ, each conditioned on the observed noisy* FM *signal over $(0, t)$. The former quantity can be used to minimize the probability of error in guessing—from the observations over $(0, t)$—whether τ has occurred by t.*

I. INTRODUCTION

The theory of frequency modulation has always been beset by analytical difficulties, and nowhere have these been more in evidence than in the area of optimal demodulation of noisy FM signals. Recent advances in nonlinear filtering, however, make it possible to solve certain problems of detection and estimation quite explicitly. We report on such a class of problems here.

The basic problem setup is this: an event of interest occurs at a random time τ. Its occurrence is signaled by sending a pulse of shape $h(\cdot)$, starting at τ; that is, we send

$$s(t) = \begin{cases} 0 & t < \tau \\ h(t-\tau) & t \geq \tau \end{cases},$$

where $h(\cdot)$ is some causal, integrable pulse. The signal $s(t)$ is transmitted by FM; the waveform is

$$\cos\left[\theta + \omega t + \int_0^t s(u)du\right]$$

for a carrier frequency ω and initial phase θ. In transmission this wave suffers the degradation of having white noise added to it; thus, we observe a signal y_t defined by

$$dy_t = \cos\left[\theta + \omega t + \int_0^t s(u)du\right]dt + db_t,$$

with b_t a Brownian motion independent of τ. We would like to construct a nonlinear filter acting causally on y_t to estimate optimally at each time t whether $\tau < t$ or not, and if so, by how much. This filter will be obtained by solving the nonlinear filtering problem of determining the conditional probability

$$p_0(t) = P\{\tau > t \mid y_s, 0 \leq s \leq t\}$$

and the conditional density (u = distance back from t to τ)

$$p_1(t, u) = P\{\tau \epsilon d(t-u) \mid y_s, 0 \leq s \leq t\}, \quad 0 \leq u \leq t.$$

Such a filter (p_0, p_1) represents a summary, without loss, of all the information in the "past" $\sigma\{y_s, 0 \leq s \leq t\}$ that is relevant to whether τ occurred by time t, and if so, how far back. In particular, the filter (p_0, p_1) yields least-squares estimates of τ, by integration over u, according to the formula

$$E\{\tau \mid y_s, 0 \leq s \leq t\} = p_0(t)\frac{\int_t^\infty uf(u)du}{1 - F(t)} + \int_0^t (t-u)p_1(t, u)du,$$

where F is the a priori distribution of τ, and $f = F'$ its density. The first term predicts where τ will be, on the average, when it has not yet occurred by t; the second "postdicts" τ when it has already happened by time t. Indeed, the first term is $E\{\tau 1_{\tau > t} \mid y_s, 0 \leq s \leq t\}$ and the second is $E\{\tau 1_{\tau \leq t} \mid y_s, 0 \leq s \leq t\}$.

II. NOTATIONS

Let x_t be the process $1_{\tau \leq t}$ so that

$$x_t = \begin{cases} 0 \text{ if the event has not occurred by time } t. \\ 1 \text{ if the event occurred by time } t. \end{cases}$$

Then with $X_t = \int_0^t x_s ds$, the signal $s(t)$ can be written as

$$s(t) = \int_0^t h(t-s)dx_s = \begin{cases} H(X_t) & t \geq \tau \\ 0 & t < \tau \end{cases}$$

and the FM signal as

$$\cos[\theta + \omega t + H(X_t)],$$

where $H = \int_0 h(s)ds$.

III. FILTERING EQUATIONS

Our approach is Bayesian: foreknowledge of distr $\{\tau\}$ is used to calculate the conditional probabilities p_0 and p_1. We assume for simplicity, and with only slight loss of generality, that τ has a known a priori distribution F with a differentiable density f. The "rate" at which τ is occurring during $(t, t+h)$, given that it has not yet happened, is just

$$\lambda(t) = \frac{f(t)}{1-F(t)}, > 0.$$

We assume at first that the phase θ is known at the receiver. Then, the filtering or Zakai equations for unnormalized versions ρ_0 and ρ_1 of p_0 and p_1, respectively, are just

$$d\rho_0 = \lambda(t)\rho_0 dt + \cos(\theta + \omega t)\rho_0 dy_t,$$

$$d\rho_1 = \frac{\partial \rho_1}{\partial u} + \cos[\theta + \omega t + H(u)]\rho_1 dy_t,$$

with initial conditions $\rho_0(0) = 1$,

$$\lim_{t \downarrow 0} \int_0^t \rho_1(t, u)du = 0$$

and boundary condition $\rho_1(t, 0) = \rho_0(t)\lambda(t)$.

It can be seen that since the process x_t is transient in character these equations are coupled in one direction only: ρ_1 depends on ρ_0 via the boundary condition, but ρ_0 in no way depends on ρ_1. Thus, it will be possible to solve for ρ_0 first, and then for ρ_1. We first transform the problem into one without stochastic differentials. This is done by the now familiar device[1] of looking for a solution of the form

$$\rho_0(t) = \exp[y_t \cos(\theta + \omega t)]q_0(t)$$

$$\rho_1(t, u) = \exp\{y_t \cos[\theta + \omega t + H(u)]\}q_1(t, u), \quad 0 \le t \le u,$$

where q_0 and q_1 are differentiable functions, though not necessarily C^1. This form for ρ_0 and ρ_1 indicates that the rough or martingale dependence of these functions on $y(\cdot)$ is confined to the exponent as shown, while their dependence on $y(\cdot)$ via q_0 and q_1 is of a much smoother integrated form, as will be seen.

Applying Ito's formula to the postulated form, with quadratic variation $d\langle y \rangle_t = dt$ since the observation process is a translation of the Wiener process, we find these nonstochastic PDEs for q_0 and q_1:

$$\dot{q}_0 = q_0 \left(-\lambda(t) + \omega y_t \sin(\theta + \omega t) - \frac{1}{2}\cos^2(\theta + \omega t) \right), \; q_0(0) = 1$$

$$\frac{\partial q_1}{\partial t} = -\frac{\partial q_1}{\partial u} + q_1 \left(y_t[\omega + h(u)]\sin[\theta + \omega t + H(u)] - \frac{1}{2}\cos^2[\theta + \omega t + H(u)] \right), \; 0 \le u \le t.$$

The boundary condition is

$$q_1(t,0) = q_0(t)\lambda(t)\,.$$

The first equation is an ODE solvable as

$$\begin{aligned} q_0(t) &= \exp\left(-\int_0^t \lambda(s)ds + \int_0^t \left[\omega y_s \sin(\theta+\omega s) - \frac{1}{2}\cos^2(\theta+\omega s)\right]ds\right) \\ &= [1-F(t)]\exp\left(\int_0^t \left[\omega y_s \sin(\theta+\omega s) - \frac{1}{2}\cos^2(\theta+\omega s)\right]ds\right). \end{aligned}$$

The second is a first-order PDE solvable by characteristics as

$$\begin{aligned} q_1(t,u) = A(t-u)\exp\Bigg(\int_0^t \{\omega y_s \sin[\theta+\omega s + H(s-t+u)] \\ - \frac{1}{2}\cos^2[\theta+\omega s + H(s-t+u)] \\ + y_s h(s-t+u)\sin[\theta+\omega s+H(s-t+u)]\}ds\Bigg), \end{aligned}$$

where $A(\cdot)$ is an arbitrary function. To obtain A we let $u\downarrow 0$, and we use $h(s-t+u)=0$ and $H(s-t+u)=0$ for $s \le t-u$ to find

$$\begin{aligned} q_1(t,0) &= A(t)\exp\left(\int_0^t \left[y_s\omega \sin(\theta+\omega s) - \frac{1}{2}\cos^2(\theta+\omega s)\right]ds\right), \\ &= q_0(t)\lambda(t), \end{aligned}$$

by the boundary condition. Thus, $A(t) = f(t)$, and we obtain

$$\begin{aligned} q_1(t,u) = f(t-u)\exp\Bigg(\int_0^{t-u} \left[\omega y_s \sin(\theta+\omega s) - \frac{1}{2}\cos^2(\theta+\omega s)\right]ds \\ + \int_{t-u}^t \{[\omega + h(s-t+u)]y_s \sin[\theta+\omega s+H(s-t+u)] \\ - \frac{1}{2}\cos^2([\theta+\omega s+H(s-t+u)]\}ds\Bigg). \end{aligned}$$

We remark that this is the unconditional density $f(t-u)$ that τ occur at $t-u$, multiplied by a positive factor depending on the pulse shape h and the observation $\{y_s,\ 0 \le s \le t\}$. The $\cos^2$ integral can be evaluated explicitly, leading to some simplification, and to approximate formulas for large carrier frequencies.[2]

The normalization

$$p_0(t) + \int_0^t p_1(t,u)du = 1$$

is achieved by dividing each of ρ_0 and ρ_1 by

$$\rho_0(t) + \int_0^t \rho_1(t, u)du,$$

where

$$\rho_0(t) = [1 - F(t)]\exp\left(y_t\cos(\theta + \omega t) + \int_0^t [\omega y_s\sin(\theta + \omega s) - \frac{1}{2}\cos^2(\theta + \omega s)]ds \right)$$

$$\rho_1(t, u) = f(t - u)\exp\Bigg[y_t\cos[\theta + \omega t + H(u)] + \int_0^{t-u} [\omega y_s\sin(\theta + \omega s) - \frac{1}{2}\cos^2(\theta + \omega s)]ds + \int_{t-u}^t \Bigg[[\omega + h(s - t + u)]y_s\sin[\theta + \omega s + H(s - t + u)] - \frac{1}{2}\cos^2[\theta + \omega s + H(s - t + u)] \Bigg) ds \Bigg].$$

If, as is likely, the phase θ is not known at the receiver, then it must be integrated out in both ρ_0 and ρ_1 prior to normalization, a process that mars the relatively neat formulas obtained for ρ_0 and ρ_1 for θ known. With θ uniform over $(-\pi, \pi)$ and independent of τ, familiar Bessel function approximations again arise.[2]

IV. HAS τ OCCURRED YET? THE OPTIMAL GUESS

In the kind of system under study here, a task of primary interest is to guess at t whether τ has happened yet. Such a guess is represented mathematically by a random process v_t, taking the value 1 for a decision that τ has not occurred, and a value 0 for a decision that it has, and adapted to the past observations $\rho\{y_s, 0 \le s \le t\}$. The probability of error is just

$$P\{\tau \le t \;\&\; v_t = 1\} + P\{\tau > t \;\&\; v_t = 0\},$$

which can be written as

$$\begin{aligned} &E1_{\tau>t}(1 - v_t) + Ev_t(1 - 1_{\tau>t}) \\ &= E1_{\tau>t} - 2E1_{\tau>t}v_t + Ev_t \\ &= E(1_{\tau>t} - v_t)^2, \end{aligned}$$

the mean square error in approximating $1_{\tau>t}$ by v_t. Thus, the chance of error is the least if v_t is chosen to minimize this mean square error. Noting that $p_0(t) = E\{1_{\tau>t} | y_s, 0 \le s \le t\}$, we can write this mean square error as

$$E\{p_0(t) - 2p_0(t)v_t + v_t\}$$

and conclude that a minimizing v_t is

$$v(t) = \begin{cases} 1 & \text{if } p_0(t) > \frac{1}{2} \\ 0 & \text{if } p_0(t) \le \frac{1}{2}. \end{cases}$$

It follows that by watching $p_0(\cdot)$ we can make a best guess as to whether τ has occurred yet or not, best in the sense of minimizing the chance of being wrong.

V. THE CONDITIONAL EXPECTATION OF τ

As we observe the signal y_t, we may be interested in predicting τ on the basis of the information seen so far. More precisely, since it is possible that at $t > 0$ τ has already occurred, we want to simultaneously predict and "postdict" τ by calculating the two terms in

$$\begin{aligned}\hat{\tau} &= E\{\tau | y_s, 0 \le s \le t\} \\ &= E\{\tau 1_{\tau>t} | y_s, 0 \le s \le t\} + E\{\tau 1_{\tau\le t} | y_s, 0 \le s \le t\}.\end{aligned}$$

The second term is clearly given in terms of p_1 by

$$\int_0^t (t-u)p_1(t,u)du, \quad (u = \text{distance back to } \tau \text{ from } t).$$

We claim that the first is just

$$p_0(t)\frac{\displaystyle\int_t^\infty u dF(u)}{1 - F(t)}.$$

For with $\sigma\{y_s, 0 \le s \le t\} = Y_0^t$ for short, we have

$$\begin{aligned}E\{\tau 1_{\tau>t} | y_0^t\} &= E\{\tau 1_{\tau>t} | y_0^t \cup \tau > t\} | y_0^t\} \\ &= E\{E\{\tau | y_0^t \cup \tau > t\} 1_{\tau>t} | y_0^t\} \\ &= E\{E\{\tau | \tau > t\} 1_{\tau>t} | y_0^t\} \\ &= p_0(t)E\{\tau | \tau > t\}\end{aligned}$$

since the additional y_s information in $Y_0^t \cup \tau > t$ is irrelevant to τ when it is known that $\tau > t$. That is, since $x_t = 1_{\tau>t}$ is a Markov process, all the information $\sigma\{x_s, y_s, 0 \le s \le t\}$ is irrelevant to $\{x_u, u > t\}$ when it is known that $x_t = 1$, i.e., $\tau > t$.

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